Chapter 0  General Mathematics and Mathematical Methods

1. Indices
For any real numbers x, a and b: \[ x^0 = 1 \]
\[ x^b = x \times x \times \cdots \times x \text{ b times} \]
\[ x^{-a} = \frac{1}{x^a} \]
\[ x^{a/b} = \sqrt[b]{x^a} \]
\[ x^a \times x^b = x^{a+b} \]
\[ (x^a)^b = x^{ab} \]

2. Logarithm
For any positive real numbers a, x and y: \[ \log_a (x \cdot y) = \log_a x + \log_a y \]
\[ \log_a (x)^y = y \cdot \log_a x \]
\[ \log_y x = \frac{\log_a x}{\log_a y} \]
For \( y = \log_a x \) - if \( a = 10 \), it is called the common logarithm, and usually denoted simply by \( y = \log x \). \( (x = 10^y) \)
- if \( a = e \), it is called the natural logarithm, and usually denoted by \( y = \ln x \). \( (x = e^y) \)
- Whenever \( x = a \), the result is equal to 1.

3. Trigonometry
\[ \sin \theta = \frac{o}{h} ; \quad \cos \theta = \frac{a}{h} \]
\[ \tan \theta = \frac{o}{a} = \frac{\sin \theta}{\cos \theta} \]
\[ h^2 = a^2 + o^2 \]
\[ \sin^2 \theta + \cos^2 \theta = 1 \]
Most of the time, the unit of an angle is in radian rather than in degree. The relationship between them is given by \( \theta \) (in radian) = \[ \frac{\pi}{180^\circ} \cdot \theta \) (in degree)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \pi )</th>
<th>( \frac{3\pi}{2} )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
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<tr>
<td>cos</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( -\frac{\sqrt{3}}{2} )</td>
<td>( -\frac{\sqrt{2}}{2} )</td>
<td>( -1 )</td>
</tr>
<tr>
<td>tan</td>
<td>0</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>1</td>
<td>( \sqrt{3} )</td>
<td>( -\sqrt{3} )</td>
<td>( -\frac{1}{\sqrt{3}} )</td>
<td>0</td>
<td>( \frac{1}{\sqrt{3}} )</td>
</tr>
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</table>

\[ \sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B \]
\[ \sin(2A) = 2 \sin A \cdot \cos A \]
\[ \cos(A \pm B) = \cos A \cdot \cos B \pm \sin A \cdot \sin B \]
\[ \cos(2A) = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A \]
\[ \sin^2 \frac{A}{2} = \frac{1}{2} (1 - \cos A) \]
\[ \cos^2 \frac{A}{2} = \frac{1}{2} (1 + \cos A) \]
4. Approximations: - \((1 \pm x)^n = \sum_{r=0}^{n} \binom{n}{r} (\pm x)^{n-r}\), for \(x \to 0\), \((1 \pm x)^n \approx 1 \pm nx\)

- \((1 \pm x)^n \approx \frac{(1 \mp x)^n}{(1 - x^2)^n} \approx 1 \pm nx\) when \(x \to 0\)
- For small \(\theta\) (in radian), \(\sin \theta \approx \tan \theta \approx \theta\) and \(\cos \theta \approx 1\)

5. Inequality
For any positive real numbers \(a\), \(b\) and \(c\):
(i) If \(a \geq b\), \(a\) is minimum when \(a = b\)
(ii) If \(a \leq b\), \(a\) is maximum when \(a = b\)
(iii) If \(a = b\), \(a \leq b + c\)

6. Graphs
- \(y = mx + c\) represents a straight line with
  (i) slope = \(m\);
  (ii) y-intercept = \(c\) (when \(x = 0\));
  (iii) x-intercept = \(-\frac{c}{m}\) (when \(y = 0\)).

- \(y = mx\) represents a straight line passing through the origin.
For the above two cases, \(y\) is proportional to \(x\).
- \(y = c\) is a straight line parallel to \(x\)-axis and \(y\) is independent of \(x\);
- \(x = c\) is a straight line parallel to \(y\)-axis and \(x\) is independent of \(y\).
- If \(y\) is inversely proportional to \(x\), i.e. \(y = \frac{k}{m}\) where \(k\) is the constant of proportionality; the corresponding shape of the graph as shown on the right.
- \(y = ax^2 + bx + c\) is the equation of a parabola. If \(a\) is a positive (negative) real number, the curve is concave upward (downward) with a minimum (maximum) value of \(y = c - \frac{b^2}{4a}\) when \(x = -\frac{b}{2a}\).

- Below are the graphs that related to the exponential function

- To verify a law by reducing the expression into a form of a straight line.
  e.g. \(r^3 = \frac{GM}{4\pi^2} T^2\), a straight line can be produced by plotting \(r^3\) against \(T^2\) with \(\frac{GM}{4\pi^2}\) as the slope.
- To find the relationship between \(x\) and \(y\) which are in the form of \(y = ax^b\), we should plot \(\log y\) versus. \(\log x\), then a straight line is obtained with (i) slope = \(b\) and (ii) intercept = \(\log a\).

7. Differentiation, \(\frac{dy}{dx}\)
- \(\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}\) (= the derivative of \(y\)) = slope of the tangent to a curve
e.g. the rate of change of momentum is represented by \( F = \frac{d(mv)}{dt} \)

- \( \frac{d}{dx} \) (constant) = 0

For \( n \) and \( k \) are constants:
- \( \frac{d}{dx} (k \cdot x^n) = kn \cdot x^{n-1} \)
- \( \frac{d}{dx} [\sin(kx)] = k \cdot \cos(kx) \)
- \( \frac{d}{dx} [\cos(kx)] = -k \cdot \sin(kx) \)
- \( \frac{d}{dx} [k \cdot \ln(nx)] = \frac{k}{x} \)
- \( \frac{d}{dx} (k \cdot e^{nx}) = kn \cdot e^{nx} \)

- We can use the differentiation technique to find the extremal.
  e.g. With reference to the equation of a parabola, \( y = ax^2 + bx + c \), the value of \( x \) such that \( y \) attains its maximum or minimum can be found by setting \( \frac{dy}{dx} = 0 \) i.e. at \( x = -\frac{b}{2a} \).

- There exist some quantities that the second derivatives \( \frac{d^2}{dx^2} \) have meanings. e.g. We get the acceleration by differentiating displacement w.r.t. time twice, i.e. \( a = \frac{d^2s}{dt^2} = \ddot{s} \).

- Chain rule: If \( y \) and \( z \) are functions of \( x \), then \( \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \)

8. Integration, \( \int dx \) (the reverse process of differentiation)
\[
\int_a^b y \, dx = \lim_{n \to \infty} \sum_{i=0}^{n} k \cdot y_k \quad \text{where} \quad k = a + i \cdot \frac{b - a}{n} \quad \text{which is the area under the curve.}
\]
e.g. The area under the velocity-time graph (non-uniform) represents the displacement within the time interval (\( t_1 \) to \( t_2 \)) that is denoted by \( s = \int_{t_1}^{t_2} v \, dt \)

For \( n \), \( k \) and \( C \) are constants (where \( C \) can be found by substituting the initial conditions):
- \( \int (k \cdot x^n) \, dx = \frac{k}{n+1} x^{n+1} + C \)
- \( \int k \cdot \sin(nx) \, dx = -\frac{k}{n} \cos(nx) + C \)
- \( \int k \cdot \cos(nx) \, dx = \frac{k}{n} \sin(nx) + C \)
- \( \int_{0}^{2\pi} k \cdot \sin x \, dx = 0 \)
- \( \int_{0}^{2\pi} k \cdot \cos x \, dx = 0 \)
- \( \int k \cdot e^{nx} \, dx = \frac{k}{n} e^{nx} + C \)
- \( \int \frac{1}{x} \, dx = \ln x + C \)
9. Vectors
- Vector addition (head to tail) is independent of the order of addition.
  (subtraction is equivalent to addition but reversing the vector)
  \[ \mathbf{a} - \mathbf{c} + \mathbf{b} = \mathbf{a} + \mathbf{b} + (-\mathbf{c}) \]

This is similar to the parallelogram law: The diagonal of the parallelogram drawn from
the point at which two forces, represented by the sides of the parallelogram, acting on gives the
direction and magnitude of the resultant force.

- Resolution of vector (Reverse process of vector addition)
  For any given vector, we can resolve it into two coplanar vectors. Most often, these two
  vectors are mutually perpendicular \( \Rightarrow \) they are independent.

- Relative Vectors
  Definition: A vector \( \mathbf{v}_1 \) relative to another vector \( \mathbf{v}_2 \) = \( \mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2 \)
  Its physical meaning is that assuming you are with \( \mathbf{v}_2 \) (i.e. staying at rest) and
  looking at \( \mathbf{v}_1 \).

- Vector Multiplication
  For two vectors \( \mathbf{a} = a_i \mathbf{i} + a_j \mathbf{j} + a_k \mathbf{k} \) and \( \mathbf{b} = b_i \mathbf{i} + b_j \mathbf{j} + b_k \mathbf{k} \)
  (i) Dot product / scalar product (2-D)
    \[ \mathbf{i} \cdot \mathbf{j} = j \cdot i = 1 \quad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 0 \]
    \[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta \]
    e.g. Work done equals the dot product between the force, \( \mathbf{F} \) and the displacement, \( \mathbf{s} \)
    \( (W = \mathbf{F} \cdot \mathbf{s} ) \)
  (ii) Cross product / vector product (3-D) - obeying the right-hand screw rule
    \[ \mathbf{i} \times \mathbf{j} = \mathbf{k} ; \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} ; \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \]
    \[ \mathbf{j} \times \mathbf{i} = -\mathbf{k} ; \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} ; \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \]
    \[ \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \]
    \[ \mathbf{a} \times \mathbf{b} = (|\mathbf{a}| |\mathbf{b}| \sin \theta) \mathbf{n} \]
    where \( \mathbf{n} \) is an unit vector that is perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \).
    \[ \mathbf{a} \times \mathbf{b} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{array} \right| = (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k} \]
    e.g. The induced force, \( \mathbf{F} \) experienced by a moving charge, \( q \) with a velocity, \( \mathbf{v} \) in a
    magnetic field, \( \mathbf{B} \) \( (\mathbf{F} = q \mathbf{v} \times \mathbf{B}) \)

10. Dimensional Analysis
- Fundamental Quantities:- mass (kg), length (m) and time (t), amount of substance (mol),
  temperature (K), electric current (A), luminous intensity (cd).

- Dimensions
  For any physical quantity, the dimension can be expressed as \([M^a L^b T^c N^d \theta^e (EC)^f (LI)^g]\)
  (when all \( a, b, c, d, e, f \) and \( g \) equal zero, the quantity is said to be dimensionless)

Ex. \( (P + \frac{a}{V^2})(V - b) = nRT \), find the dimensions of \( a \), \( b \) and \( R \) with \( n = \) number of moles
- Analysis

With the help of dimension checking, we can deduce some useful formulae.

Ex. The period \( T \) of a simple pendulum undergoing small oscillations is given by

\[
T = K m^x l^y g^z
\]

where
- \( K \) - dimensionless constant
- \( m \) - mass of the bob
- \( g \) - acceleration due to gravity
- \( l \) - length of pendulum

Find the values of \( x \), \( y \) and \( z \). Hence write a correct expression for \( T \).